Asymptotics of Young tableaux in the strip, the d-sums

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Abstract. The asymptotics of the "strip" sums $S_\ell^{(\alpha)}(n)$ and of their d-sums generalizations $T_{d,ds}^{(\alpha)}(dm)$ (see Definition 1.1) were calculated in [5]. It was recently noticed that when $d > 1$ there is a certain confusion about the relevant notations in [5], and the constant in the asymptotics of these d-sums $T_{d,ds}^{(\alpha)}(dm)$ seems to be off by a certain factor. Based on the techniques of [5] we again calculate the asymptotics of the d-sums $T_{d,ds}^{(\alpha)}(dm)$. We do it here carefully and with complete details. This leads to Theorem 1.2 below, which replaces Corollary 4.4 of [5] in the cases $d > 1$.

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1 Introduction

Let λ be a partition and $\ell(\lambda)$ the number of non-zero parts of λ. Let f^λ denote the number of standard tableaux of shape λ. For the Young-Frobenius formula for f^λ see for example [2, 2.3.22], and for the *"hook"* formula see for example [8, corollary 7.21.5].

The asymptotics of the sums $S_\ell^{(\alpha)}(n)$ and of the d-sums $T_{d,ds}^{(\alpha)}(dm)$ (see Definition 1.1) were studied in [5], see [5, Corollary 4.4] (there we used the notation d_λ instead of f^λ). We recently noticed that when $d > 1$ there is a certain confusion about the notations in [5], and the constant in the asymptotics of the d-sums $T_{d,ds}^{(\alpha)}(dm)$ seems to be off by a certain factor.

Based on the techniques of [5] we calculate, with complete details, the asymptotics of the d-sums $T_{d,ds}^{(\alpha)}(dm)$. While ths asymptotic formula for the sums $S_\ell^{(\alpha)}(n)$ remain unchanged as in [5], this leads to a new asymptotic formula for the d-sums $T_{d,ds}^{(\alpha)}(dm)$, given in Theorem 1.2 below.

The validity of Theorem 1.2 can be tested as follows. In few cases the d-sums $T_{d,ds}^{(\alpha)}(dm)$ are given by a closed formula, which yield the corresponding asymptotics directly– independent of Theorem 1.2. In all these cases, the direct asymptotics and the asymptotics deduced from Theorem 1.2 – agree, see Section 3.1. Also, for small values of d and s it is possible to write an explicit formula for, say, $T_{d,ds}^{(1)}(dm)$. By Theorem 1.2 $T_{d,ds}^{(1)}(dm) \simeq A(d, s, dm)$.

Now form the ratio $T^{(1)}_{d,ds}(dm)/A(d,s,dm)$. Using, say, "Mathematica", calculate that ratio for increasing values of m, verifying that these values become closer and closer to 1 as m increases. This again tests and indicates the validity of Theorem 1.2.

1.1 The main theorem

The following definition recalls the d-sums from [5].

Definition 1.1. Let $m, s, d \geq 1$, then define

1. $$B_d(dm) = \{\lambda \vdash dm \mid d \quad \text{divides all} \quad \lambda'_j\}.$$
   Note that $\lambda \in B_d(dm)$ if and only if λ can be written as $\lambda = (\mu_1^d, \mu_2^d, \ldots)$ with $(\mu_1, \mu_2, \ldots) \vdash m$, and then d divides $\ell(\lambda)$.

2. $$B_{d,ds}(dm) = \{\lambda \in B_d(dm) \mid \ell(\lambda) \leq ds\} \qquad \text{and}$$

3. $$T^{(\alpha)}_{d,ds}(dm) = \sum_{\lambda \in B_{d,ds}(dm)} (f^\lambda)^\alpha.$$

4. When $d = 1$ we denote $T^{(\alpha)}_{1,s}(m) = S^{(\alpha)}_s(m)$. Thus
   $$S^{(\alpha)}_s(m) = \sum_{\lambda \vdash m,\ \ell(\lambda) \leq s} (f^\lambda)^\alpha.$$

We correct [5, Corollary 4.4] in the case $d > 1$ by proving the following theorem (see Theorem 3.3 below). Here the variable N is replaced by s.

Theorem 1.2. *Let $1 \leq d, s \in \mathbb{Z}$ and let $0 < \alpha \in \mathbb{R}$. As $m \to \infty$,*

$$T^{(\alpha)}_{d,ds}(dm) \simeq$$

$$\simeq \left[\left(\frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot s^{d^2 s^2/2} \cdot (2! \cdots (d-1)!)^s \cdot \left(\frac{1}{\sqrt{m}} \right)^{(d^2 s^2 + d^2 s - 2)/2} \cdot (ds)^{dm} \right]^\alpha \cdot$$

$$\cdot (\sqrt{m})^{s-1} \cdot \left(\frac{d}{s} \right)^{(s-1)(\alpha s + 2)/4} \cdot \frac{d}{\sqrt{s}} \cdot \sqrt{\frac{\alpha}{2\pi}} \cdot \frac{1}{s!} \cdot$$

$$\cdot (2\pi)^{s/2} \cdot (d^2\alpha)^{-s/2 - d^2 \alpha s(s-1)/4} \cdot (\Gamma(1 + d^2\alpha/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1 + d^2 \alpha j/2).$$

2 Asymptotics for a single f^λ

The following proposition corrects (and replaces) [5, (F.1.3)], and is the key for proving Theorem 1.2. Recall the notation

$$D_s(x_1,\dots,x_s)=\prod_{1\le i<j\le s}(x_i-x_j).$$

Proposition 2.1. *Let $\lambda=(\lambda_1^d,\dots,\lambda_s^d)\vdash dm=n$. For $1\le i\le s$ write $\lambda_i=m/s+b_i\sqrt{m}$ and assume the b_i are bounded, so $\lambda_i\simeq m/s$. Then, as m goes to infinity,*

$$f^\lambda\simeq\left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1}\cdot\sqrt{d}\cdot s^{d^2s^2/2}\cdot(2!\cdots(d-1)!)^s\cdot\left(\frac{1}{\sqrt{m}}\right)^{(d^2s^2+d^2s-2)/2}\cdot(ds)^{dm}\cdot$$
$$\cdot D_s(b_1,\dots,b_s)^{d^2}\cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}=$$
$$=\left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1}\cdot d^{(d^2s^2+d^2s)/4}\cdot s^{d^2s^2/2}\cdot(2!\cdots(d-1)!)^s\cdot\left(\frac{1}{\sqrt{dm}}\right)^{(d^2s^2+d^2s-2)/2}\cdot(ds)^{dm}\cdot$$
$$\cdot D_s(b_1,\dots,b_s)^{d^2}\cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}.$$

Proof. Apply, for example, the Young-Frobenius formula for f^λ: First, all $\lambda_i\simeq m/s$, hence we can write

$$f^\lambda\simeq\left(\frac{s}{m}\right)^{ds(ds-1)/2}\cdot\frac{(dm)!}{(\lambda_1!)^d\cdots(\lambda_s!)^d}\cdot H(\lambda_1,\dots,\lambda_s)\tag{1}$$

where $H(\lambda_1,\dots,\lambda_s)$ is the product of factors of the form $\lambda_i-\lambda_j+k$, with various $0\le k\le ds$, and which we now analyze.

For $1\le i<j\le s$ there are d^2 factors of f^λ of the form $\lambda_i-\lambda_j+k$, with various k's, all of them satisfying $\lambda_i-\lambda_j+k\simeq(b_i-b_j)\sqrt{m}$. The number of pairs (i,j) where $1\le i<j\le s$ is $s(s-1)/2$, and each such pair contributes d^2 times the factor $(b_i-b_j)\sqrt{m}$, hence the factor $D_s(b_1,\dots,b_s)^{d^2}\cdot(\sqrt{m})^{d^2s(s-1)/2}$ in (2) below.

In the cases $i=j$ each of the s blocks (λ_i^d) contributes $D_d(d,d-1,\dots,1)=1!\cdot2!\cdots(d-1)!$, hence the factor $(1!\cdot2!\cdots(d-1)!)^s$ in (2) below. It follows that

$$f^\lambda\simeq\left(\frac{s}{m}\right)^{ds(ds-1)/2}\cdot(2!\cdots(d-1)!)^s\cdot D_s(b_1,\dots,b_s)^{d^2}\cdot(\sqrt{m})^{d^2s(s-1)/2}\cdot\frac{(dm)!}{(\lambda_1!)^d\cdots(\lambda_s!)^d}.\tag{2}$$

Again since $\lambda_i\simeq m/s$,

$$\frac{m!}{(\lambda_1+s-1)!\cdots(\lambda_s)!}\simeq\left(\frac{s}{m}\right)^{s(s-1)/2}\cdot\frac{m!}{(\lambda_1!)\cdots(\lambda_s!)}.\tag{3}$$

By [5, Step 3, page 118, with $\sqrt{2\pi}$ replacing and correcting 2π]

$$\frac{m!}{(\lambda_1+s-1)!\cdots(\lambda_s)!} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s^2/2} \cdot \left(\frac{1}{m}\right)^{(s^2-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2+\cdots+b_s^2)}, \tag{4}$$

hence by (3) and (4)

$$\frac{m!}{(\lambda_1!)\cdots(\lambda_s!)} \simeq \left(\frac{m}{s}\right)^{s(s-1)/2} \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s^2/2} \cdot \left(\frac{1}{m}\right)^{(s^2-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2+\cdots+b_s^2)} =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s/2} \cdot \left(\frac{1}{m}\right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2+\cdots+b_s^2)}. \tag{5}$$

Now

$$\frac{(dm)!}{(\lambda_1!)^d\cdots(\lambda_s!)^d} \simeq \frac{(dm)!}{(m!)^d} \cdot \left(\frac{m!}{\lambda_1!\cdots\lambda_s!}\right)^d \tag{6}$$

and by Stirling's formula

$$\frac{(dm)!}{(m!)^d} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{d-1} \cdot \sqrt{d} \cdot \left(\frac{1}{\sqrt{m}}\right)^{d-1} \cdot d^{dm}. \tag{7}$$

It follows from (5), (6) and (7) that

$$\frac{(dm)!}{(\lambda_1!)^d\cdots(\lambda_s!)^d} \simeq$$

$$\simeq \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{d-1} \cdot \sqrt{d} \cdot \left(\frac{1}{\sqrt{m}}\right)^{d-1} \cdot d^{dm}\right] \cdot \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s/2} \cdot \left(\frac{1}{m}\right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2+\cdots+b_s^2)}\right]^d =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1} \cdot \sqrt{d} \cdot s^{ds/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{ds-1} \cdot (ds)^{dm} \cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}. \tag{8}$$

Together with (2) this yields

$$f^\lambda \simeq \left[\left(\frac{s}{m}\right)^{ds(ds-1)/2} \cdot D_s(b_1,\ldots,b_s)^{d^2} \cdot (2!\cdots(d-1)!)^s \cdot (\sqrt{m})^{d^2 s(s-1)/2}\right] \cdot$$

$$\cdot \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1} \cdot \sqrt{d} \cdot s^{ds/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{ds-1} \cdot (ds)^{dm} \cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}\right] =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1} \cdot \sqrt{d} \cdot s^{d^2s^2/2} \cdot (2!\cdots(d-1)!)^s \cdot \left(\frac{1}{\sqrt{m}}\right)^{(d^2s^2+d^2s-2)/2} \cdot (ds)^{dm} \cdot$$

$$\cdot D_s(b_1,\ldots,b_s)^{d^2} \cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}.$$

This completes the proof of Proposition 2.1. $\square$

2.1 Some examples

Example 2.2. Using "Mathematica", Proposition 2.1 was tested and confirmed in the case $d = 3,\ s = 2,\ b_1 = 1$ and $b_2 = -1$, and with $n = 3m$ getting larger and larger.

Example 2.3. The case $s = 1$, any d, so $\lambda = (m, \dots, m) = (m^d)$. In this case

$$f^\lambda = \frac{(dm)! \cdot 2! \cdots (d-1)!}{m! \cdot (m+1)! \cdots (m+d-1)!}.$$

By applying Stirling's formula directly we get that as $m \to \infty$,

$$f^\lambda \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{d-1} \cdot 2! \cdots (d-1)! \cdot \sqrt{d} \cdot \left(\frac{1}{\sqrt{m}}\right)^{d^2-1} \cdot d^{dm}.$$

This agrees with Proposition 2.1 since the factor $D_s(b_1, \dots, b_s)^{d^2} \cdot e^{-(ds/2)(b_1^2+\cdots+b_s^2)}$ in that proposition equals 1 in this case.

Example 2.4. Here we repeat the proof of Proposition 2.1 - in the case $d = s = 2$, showing more explicitly the various steps of the calculations. Let $\lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2) \vdash 2m$, so $(\lambda_1, \lambda_2) \vdash m$. Let $\lambda_j = \frac{m}{2} + b_j\sqrt{m} \simeq \frac{m}{2}$. In that case we verify directly that

$$f^\lambda \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{3} \cdot 2^{14} \cdot \left(\frac{1}{\sqrt{2m}}\right)^{11} 4^{2m} \cdot (b_1 - b_2)^4 \cdot e^{-2(b_1^2+b_2^2)}. \tag{9}$$

Proof. By either the hook formula or by the Young-Frobenius formula

$$f^\lambda = \frac{(2m)! \cdot (\lambda_1 - \lambda_2 + 1) \cdot (\lambda_1 - \lambda_2 + 2)^2 \cdot (\lambda_1 - \lambda_2 + 3)}{(\lambda_1 + 3)! \cdot (\lambda_1 + 2)! \cdot (\lambda_2 + 1)! \cdot \lambda_2!}.$$

Also $\lambda_i + j \simeq m/2$ while $\lambda_1 - \lambda_2 + j \simeq (b_1 - b_2)\sqrt{m}$, hence

$$f^\lambda \simeq \left(\frac{2}{m}\right)^{6} \cdot (b_1 - b_2)^4 \cdot m^2 \cdot \frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2}. \tag{10}$$

By [5, "Step 3" with $\sqrt{2\pi}$ replacing 2π (page 118)]

$$\frac{m!}{(\lambda_1 + 1)! \cdot \lambda_2} \simeq \frac{1}{\sqrt{2\pi}} \cdot 4 \cdot 2^m \cdot \left(\frac{1}{m}\right)^{3/2} \cdot e^{-(b_1^2+b_2^2)}.$$

Since $\lambda_1 + 1 \simeq m/2$,

$$\frac{m!}{\lambda_1! \cdot \lambda_2!} \simeq \frac{m}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot 4 \cdot 2^m \cdot \left(\frac{1}{m}\right)^{3/2} \cdot e^{-(b_1^2+b_2^2)} = \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot 2^m \cdot \left(\frac{1}{m}\right)^{1/2} \cdot e^{-(b_1^2+b_2^2)}.$$

Also
$$\frac{(2m)!}{(m!)^2} \simeq \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{m}} \cdot 2^{2m}.$$
Thus
$$\frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} = \left(\frac{m!}{\lambda_1! \cdot \lambda_2!}\right)^2 \cdot \frac{(2m)!}{(m!)^2} \simeq$$
$$\left[\frac{1}{\sqrt{2\pi}} \cdot 2 \cdot 2^m \cdot \left(\frac{1}{m}\right)^{1/2} \cdot e^{-(b_1^2+b_2^2)}\right]^2 \cdot \left[\frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{m}} \cdot 2^{2m}\right]$$
namely
$$\frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot 4 \cdot \sqrt{2} \cdot 4^{2m} \cdot \left(\frac{1}{m}\right)^{3/2} \cdot e^{-2(b_1^2+b_2^2)}. \tag{11}$$
Finally
$$f^\lambda \simeq \left(\frac{2}{m}\right)^6 \cdot m^2 \cdot (b_1-b_2)^4 \cdot \frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} \simeq$$
$$\simeq \left(\frac{2}{m}\right)^6 \cdot m^2 \cdot (b_1-b_2)^4 \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot 4 \cdot \sqrt{2} \cdot 4^{2m} \cdot \left(\frac{1}{m}\right)^{3/2} \cdot e^{-2(b_1^2+b_2^2)} =$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot 2^{14} \cdot \left(\frac{1}{\sqrt{2m}}\right)^{11} 4^{2m} \cdot (b_1-b_2)^4 \cdot e^{-2(b_1^2+b_2^2)},$$
which verifies (9). □

3 Asymptotics for the general sums

As in [5, Theorem 3.2], Proposition 2.1 implies

Theorem 3.1. *[5, Corollary 4.4 corrected] Let $\Omega(s) \subset \mathbb{R}^s$ denote the following domain:*
$$\Omega(s) = \{(x_1, \dots, x_s) \in \mathbb{R}^s \mid x_1 \geq \cdots \geq x_s \quad and \quad x_1 + \cdots + x_s = 0\}.$$
Also recall Definition 1.1. Then, as $m \to \infty$,
$$T^{(\alpha)}_{d,ds}(dm) \simeq$$
$$\simeq \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1} \cdot \sqrt{d} \cdot s^{d^2s^2/2} \cdot (2! \cdots (d-1)!)^s \cdot \left(\frac{1}{\sqrt{m}}\right)^{(d^2s^2+d^2s-2)/2} \cdot (ds)^{dm}\right]^\alpha \cdot$$
$$\cdot (\sqrt{m})^{s-1} \cdot I(d^2, s, \alpha)$$
where
$$I(d^2, s, \alpha) = \int_{\Omega(s)} \left[D_s(x_1, \dots, x_s]^{d^2} \cdot e^{-(ds/2)(x_1^2+\cdots+x_s^2)}\right]^\alpha \cdot dx_1 \cdots dx_{s-1}.$$

Remark 3.2. Note that by [5, Section 4] and by the Selberg integral [1], [3], [6]

$$I(d^2,s,\alpha)=\left(\frac{d}{s}\right)^{(s-1)(\alpha s+2)/4}\cdot\frac{d}{\sqrt{s}}\cdot\sqrt{\frac{\alpha}{2\pi}}\cdot\frac{1}{s!}\cdot$$
$$\cdot(2\pi)^{s/2}\cdot(d^2\alpha)^{-s/2-d^2\alpha s(s-1)/4}\cdot(\Gamma(1+d^2\alpha/2))^{-s}\cdot\prod_{j=1}^{s}\Gamma(1+d^2\alpha j/2).$$

Thus Theorem 3.1 can be rewritten as follows.

Theorem 3.3. *Let $1\le s,d\in\mathbb{Z}$ and $0<\alpha\in\mathbb{R}$. Then, as $m\to\infty$,*

$$T^{(\alpha)}_{d,ds}(dm)\simeq$$
$$\simeq\left[\left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1}\cdot\sqrt{d}\cdot s^{d^2s^2/2}\cdot(2!\cdots(d-1)!)^s\cdot\left(\frac{1}{\sqrt{m}}\right)^{(d^2s^2+d^2s-2)/2}\cdot(ds)^{dm}\right]^{\alpha}\cdot$$
$$\cdot(\sqrt{m})^{s-1}\cdot\left(\frac{d}{s}\right)^{(s-1)(\alpha s+2)/4}\cdot\frac{d}{\sqrt{s}}\cdot\sqrt{\frac{\alpha}{2\pi}}\cdot\frac{1}{s!}\cdot$$
$$\cdot(2\pi)^{s/2}\cdot(d^2\alpha)^{-s/2-d^2\alpha s(s-1)/4}\cdot(\Gamma(1+d^2\alpha/2))^{-s}\cdot\prod_{j=1}^{s}\Gamma(1+d^2\alpha j/2).$$

3.1 Some special cases

3.1.1 The case $s=1$

Let $s=1$. In that case $B_{d,d}(dm)=\{\lambda\}$ where $\lambda=(m,\dots,m)=(m^d)$. Thus, for Theorem 3.1 to hold, the product of the factors after the factor $[...]^\alpha$ should equal 1, which is easy to verify.

3.1.2 The sums $S^{(\alpha)}_s(m)$

In the case $d=1$, in the notations of [5], $T^{(\alpha)}_{1,s}(m)=S^{(\alpha)}_s(m)$, and Theorem 3.3 becomes

Theorem 3.4. *[5, Corollary 4.4]. Let $d=1$, $1\le s\in\mathbb{Z}$, $0\le\alpha\in\mathbb{R}$. Then, as $m\to\infty$,*

$$T^{(\alpha)}_{1,s}(m)=S^{(\alpha)}_s(m)\simeq$$
$$\simeq\left[\left(\frac{1}{\sqrt{2\pi}}\right)^{s-1}\cdot s^{s^2/2}\cdot\left(\frac{1}{\sqrt{m}}\right)^{(s^2+s-2)/2}\cdot s^m\right]^{\alpha}\cdot(\sqrt{m})^{s-1}\cdot\left(\frac{1}{s}\right)^{(s-1)(\alpha s+2)/4}\cdot\frac{1}{\sqrt{s}}\cdot\sqrt{\frac{\alpha}{2\pi}}\cdot\frac{1}{s!}\cdot$$
$$\cdot(2\pi)^{s/2}\cdot\alpha^{-s/2-\alpha s(s-1)/4}\cdot(\Gamma(1+\alpha/2))^{-s}\cdot\prod_{j=1}^{s}\Gamma(1+\alpha j/2).$$

This agrees with the asymptotic value of $S^{(\alpha)}_s(m)$ as given by [5, Corollary 4.4] in the case $d=1$.

3.1.3 The case $d = 1$ and $\alpha = 1$

Theorem 3.5. *Let $d = \alpha = 1$, then as $m \to \infty$,*

$$T^{(1)}_{1,s}(m) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s^2/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{(s^2+s-2)/2} \cdot s^m \cdot (\sqrt{m})^{s-1} \cdot \left(\frac{1}{s}\right)^{(s-1)(s+2)/4} \cdot \frac{1}{\sqrt{s}} \cdot \sqrt{\frac{1}{2\pi}} \cdot \frac{1}{s!} \cdot$$

$$\cdot (2\pi)^{s/2} \cdot (\Gamma(1+1/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+j/2) =$$

$$= (\sqrt{s})^{s(s-1)/2} \cdot \frac{1}{s!} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s(s-1)/2} \cdot s^m \cdot (\Gamma(1+1/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+j/2),$$

which agrees with [5, (F.4.5.1)].

3.1.4 The case $d = 1, \ \alpha = 2$

Consider the case $d = 1$ and $\alpha = 2$ (any s), then

$$T^{(2)}_{1,s}(n) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot \left(\frac{1}{\sqrt{2}}\right)^{s^2-1} \cdot (\sqrt{s})^{s^2} \cdot 2! \cdots (s-1)! \cdot \left(\frac{1}{\sqrt{n}}\right)^{s^2-1} \cdot s^{2n}.$$

For example, when $s = 2$ we have

$$T^{(2)}_{1,2}(n) \simeq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{n\sqrt{n}} \cdot 4^n.$$

In this case we know [4, page 64] that $T^{(2)}_{1,2}(n) = (2n)!/(n! \cdot (n+1)!) = C_n$, the n-th Catalan number, and by applying Stirling's formula directly, we obtain the same asymptotic value.

3.1.5 The case $s = d = 2$ and $\alpha = 1$

The case $s = d = 2$ and $\alpha = 1$. By Theorem 3.3

$$T^{(1)}_{2,4}(2m) \simeq \left[\left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot \sqrt{2} \cdot 2^8 \cdot \left(\frac{1}{\sqrt{m}}\right)^{11} \cdot (4)^{2m}\right] \cdot (\sqrt{m}) \cdot \frac{2}{\sqrt{2}} \cdot \sqrt{\frac{1}{2\pi}} \cdot \frac{1}{2} \cdot 2\pi \cdot 4^{-3} \cdot \frac{2! \cdot 4!}{2! \cdot 2!} =$$

$$= \frac{1}{\pi} \cdot 24 \cdot \left(\frac{1}{m}\right)^5 \cdot 4^{2m}.$$

Note that sequence A005700 of [7] gives the following remarkable identity:

$$T^{(1)}_{2,4}(2m) = \frac{6 \cdot (2m)! \cdot (2m+2)!}{m! \cdot (m+1)! \cdot (m+2)! \cdot (m+3)!}. \tag{12}$$

Applying Stirling's formula to the right-hand-side of (12) we obtain the same asymptotic value:

$$\frac{6 \cdot (2m)! \cdot (2m+2)!}{m! \cdot (m+1)! \cdot (m+2)! \cdot (m+3)!} \simeq \frac{1}{\pi} \cdot 24 \cdot \left(\frac{1}{m}\right)^5 \cdot 4^{2m},$$

thus verifying Theorem 3.3 in this case.

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